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D-branes on Singularities: New Quivers from Old

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Abstract

In this paper we present simplifying techniques which allow one to compute the quiver diagrams for various D-branes at (non-Abelian) orbifold singularities with and without discrete torsion. The main idea behind the construction is to take the orbifold of an orbifold. Many interesting discrete groups fit into an exact sequence $N \rightarrow G \rightarrow G/N$. As such, the orbifold \mathcal{M}/G is easier to compute as $(\mathcal{M}/N)/(G/N)$ and we present graphical rules which allow fast computation given the \mathcal{M}/N quiver.

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1 Introduction

D-branes at singularities give rise to low energy effective field theories of phenomenological interest. The idea behind geometric engineering [1, 2, 3, 4] is to look at the gauge theories that arise on branes at singularities. Another approach is to study the gauge theories that result from the intersection of branes at angles [5] (see, for example, [6, 7, 8] and references therein).

The quiver construction of Douglas and Moore [9] provides a diagrammatic tool to visualize the field content of supersymmetric gauge theories on branes at orbifold singularities, \mathbb{C}^n/G . The technique involves enumerating the irreducible representations of G and determining how the matter fields transform. In many cases, this procedure is difficult as it involves calculating the full (projective) representation theory of some group which might be large. Here we find a technique whereby the quiver can be constructed without knowing the full details of the representation theory of G . Such a method should be useful to model builders.

We present a systematic technique for obtaining quivers of low energy gauge theories corresponding to string orbifold singularities. For the simplest orbifolds, based on the Abelian discrete groups, our methods reduce to known results. They are most useful for orbifolds with discrete torsion [10] and for orbifolds involving non-Abelian discrete groups. The technique we present may also be generalized to other types of singularities.

We consider a number of examples of supersymmetric orbifolds of $\mathcal{M} = \mathbb{C}^n$ by a group G , which is a discrete subgroup of $SU(n)$. G is taken to act linearly on the coordinates of \mathbb{C}^n in a particular representation of G . Many of the discrete subgroups of $SU(n)$ are (semi-) direct product groups. Because of this, the groups fit into exact sequences of the form

$$0 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 0 \quad (1)$$

for $N \triangleleft G$ a normal subgroup of G . We are particularly interested in studying the case where the group G/N is Abelian, as here the technique we present is most directly applicable. The orbifolding procedure can be thought of as the quotient

$$\mathcal{M}/G = (\mathcal{M}/N)/(G/N), \quad (2)$$

that is, as an orbifold of an orbifold. Since $N, G/N$ are smaller groups than G , we expect that the construction of the quiver diagram for G using (2) may be simpler. Indeed, it is well known that the representation theory of a group G may be organized in terms of the representation theory of a normal subgroup.

Consider the Abelian case $G = \mathbb{Z}_m \times \mathbb{Z}_n$, for example. There are various possible quiver diagrams corresponding to this orbifold, depending on the choice of discrete torsion. It was realized that discrete torsion acts via projective representations of the orbifold group G [11, 12] (for related work, see for example [13, 14, 15, 16, 17]). In the orbifold of an orbifold approach, the discrete torsion is encoded as a choice of \mathbb{Z}_n action on the quiver of \mathcal{M}/\mathbb{Z}_m . This can also be seen as choices of monodromies of nodes in the quiver \mathcal{M}/N under the orbifold group G/N as encountered in Refs. [18, 19].

Less is known about the details of how discrete torsion acts in the non-Abelian case. An orbifold of \mathbb{C}^3 by the ordinary tetrahedral group \mathbb{E}_6 was recently analyzed [17]. Subsequently,

the authors of Ref. [20] calculated the discrete torsion for a number of non-Abelian groups and examined the ordinary dihedral groups \mathbb{D}_k in detail. We extend these results by looking at several discrete groups that fit into exact sequences with Abelian G/N .

One interesting fact that comes out of this analysis is that many orbifold theories are on the same moduli space of couplings of the low energy field theory. That is, unrelated orbifolds may give the same quiver diagram; the gauge theories differ only in their superpotentials.

The organization of our paper is as follows. In Section 2, we present the orbifold of an orbifold construction without discrete torsion and provide a simple diagrammatic prescription. Section 3 provides two introductory examples. We obtain the quivers of some non-Abelian subgroups of $SU(2)$ by employing exact sequences of the form (1), and we study the quiver of \mathbb{Z}_4 to illustrate a subtlety in our approach. In Section 4 we build in discrete torsion. We concentrate in this section on direct product groups $G_1 \times G_2$. In Section 5, we consider a number of examples which are discrete subgroups of $SU(3)$. Section 6 examines the different superpotentials that correspond to the \hat{A}_2 quiver depending upon the details of how that quiver is obtained.

2 The Construction without Discrete Torsion

To begin, we disregard the possibility of discrete torsion and focus on understanding the orbifold \mathcal{M}/G in terms of the orbifold $(\mathcal{M}/N)/(G/N)$. First, we must construct the quiver diagram of \mathcal{M}/N and then consider the action of the Abelian group G/N as an automorphism of this quiver. Later we will specialize to the case G/N abelian where the construction simplifies.

We first construct the quiver diagram of \mathcal{M}/N using standard methods [9]. The nodes are given by the irreducible representations of N , which may be deduced from the group algebra of N . The underlying vector space of the group algebra $\mathcal{A}(N)$ is that of the regular representation of N . If $g_i \in N$, then the group algebra of N consists of linear combinations of the form

$$a = \sum a_i g_i \tag{3}$$

with $a_i \in \mathbb{C}$.

Each irreducible representation determines a projector in the group algebra which belongs to the center of the algebra, and the linear span of these projectors generates the center of the group algebra itself.

The center of the algebra is straightforward to calculate. Indeed, for $\sum a_i g_i$ to commute with the generators g

$$\sum a_i g g_i g^{-1} = \sum a_i g_i, \tag{4}$$

we need that the coefficient of g_i on both sides be the same. Thus $a_i = a_j$ whenever there exists a g such that $g g_i g^{-1} = g_j$, that is, whenever g_i and g_j belong to the same conjugacy class $[g_i]$ of N . Thus the center of the group algebra $\mathcal{Z}\mathcal{A}(N)$ is generated by the conjugacy

classes of elements of the group N . The relation between the idempotents and the conjugacy classes is a discrete Fourier transform.

Consider now the group G . As N is a normal subgroup of G , then conjugation by elements of G leaves N invariant. Indeed, conjugation by an element of G induces a group automorphism of N and thus also an automorphism of the algebra $\mathcal{A}(N)$ to itself. Any automorphism of the algebra will leave the center fixed, and thus the action of G will act as a linear transformation on $\mathcal{Z}\mathcal{A}(N)$. In particular, the action will take idempotents to idempotents, so it will permute the irreducible representations of N .

If $g \in G$, then conjugation by gw with $w \in N$ will produce the same action on the conjugacy classes of N as g . Conjugation thus factors on G/N . Since the nodes of the quiver diagram denote representations of N , G/N will act on the quiver by permuting nodes. The action will also permute the arrows of the quiver with a twisting by some representation of the group G/N .

To understand the quiver for G , we must then study the irreducible representations of the group G/N , the irreducible representations of N , and the action of G/N on the quiver of N . Denote by P_k the projectors for the irreducible representations of N . We need to understand the algebra generated by P_k and the group algebra of G/N . Let σ_k be the list of elements of G/N . From the above discussion, we have

$$\sigma_k P_\ell \sigma_k^{-1} = P_{\sigma_k(\ell)}, \quad (5)$$

which is a tensor algebra twisted by the action of G/N on $\mathcal{Z}\mathcal{A}(N)$.

Any element of the algebra $\mathcal{A}(G)$ can be written in the form

$$\sum_{\ell k} a_{\ell,k} P_\ell \sigma_k, \quad (6)$$

and we want to know which linear combinations are in the center of this algebra. Elements of the center must commute with all the σ_m :

$$\sum_{\ell k} a_{\ell,k} P_\ell \sigma_k = \sum_{\ell k} a_{\ell,k} P_{\sigma_m(\ell)} \sigma_m \sigma_k \sigma_m^{-1}. \quad (7)$$

In the case where G/N is Abelian, we obtain

$$a_{i,k} = a_{\sigma_m^{-1}(i),k}. \quad (8)$$

Elements of the center must also commute with P_ℓ . We then find

$$\sum_k a_{i,k} (P_i - P_{\sigma_k(i)}) \sigma_k = 0. \quad (9)$$

Thus elements of the center can have non-zero $a_{i,k}$ only when σ_k acts trivially on P_i . For each P_k the group G/N will generate an orbit of irreducible representations. In the above result, we get one element of the center for each of these orbits and for each $e \in G/N$ which leaves the P_k fixed.

The orbit of P is a representation of the group G/N , and the elements which leave the P fixed is a (normal) subgroup of G/N . The projectors built out of the e are in one to one correspondence with the irreducible representations of this subgroup of G/N . If the orbit of P_k has d elements, we get $|G/N|/d$ irreducible representations, each of dimension $d \dim P_k$. In particular, if a P is fixed under G/N , the node associated to the representation splits into $|G/N|$ nodes, and if no element of G/N leaves P invariant, then the orbit of P contracts to a single node.

The result above determines the nodes of the new quiver diagram. We also need to find the arrows of the diagram corresponding to the matter fields. In a standard orbifold, the arrows are determined by representations of G on the normal directions to the orbifold and on the fermions. On a supersymmetric quiver these two are related, and the arrows are determined by the action of G on the directions which are transverse to the orbifold.

In our case, the quiver of N is given, and we have to understand the action of G/N on the arrows of the quiver N . If we permute the nodes, we also permute the arrows between the nodes, but there might be some extra action of G/N on the arrows.

As the original action of G on the variables normal to the orbifold is linear, we expect G/N to act linearly on the arrows of the quiver N . If two nodes are connected in the original quiver diagram, this means that the two are connected by a representation of N . N is still a subgroup of G , and the representations of G can be split into representations of N . It follows that if two orbits are connected in N , the new nodes of the two orbits are connected in some way by arrows in G .

Now, the orbifold action assigns an associated representation of G/N to each transverse field. Given a pair of nodes corresponding to two different orbits, we can decide if there is an arrow connecting them by looking at the tensor product of the corresponding representations of the subgroup of G/N with the representation associated to a transverse field.

These considerations give us the following rules.

1. The nodes of the new quiver diagram are obtained by splitting and joining nodes in the old quiver diagram according to the counting given by the orbit of the node under G/N . All the nodes in the orbit have the same rank.
2. Two orbits which are connected by arrows in the quiver of N are connected by arrows in the quiver of G .
3. If the orbits have splitting nodes, then the arrows connect nodes that are split according to the representation of G/N on the arrows.

The above rules also work if the linear action of G/N closes only up to gauge transformations. The technical point which is distinct is that we don't use representations of G/N on the arrows, but representations of the lift of G/N in $Aut(N)$, the group of automorphisms of the quiver.¹ This lift is still finite, but the group is larger than G/N , and it is not true that

¹As the arrows represent chiral fields, these are not gauge invariant, and an action of G/N on the arrows of N is only well defined up to gauge transformations. Gauge transformations correspond to inner automorphisms $Inn(N)$ of the quiver, and thus the action of G/N is an *outer* automorphism of the quiver diagram, $Out(N) = Aut(N)/Inn(N)$.

the fields are representations of G/N in general. Some still might be, whereas some others might not be.

3 New Quivers from Old

The best way to understand the application of these rules is to consider a number of examples. First, we will consider the $\widehat{\mathbb{D}}_k$ singularity. A seemingly trivial second example is provided by \mathbb{Z}_4 . However, there is an important subtlety that arises here which is important for our later discussions.

3.1 Example: $\mathbb{C}^2/\widehat{\mathbb{D}}_k$

We consider the binary dihedral group $\widehat{\mathbb{D}}_k$, which is the \mathbb{Z}_2 extension of the ordinary dihedral group \mathbb{D}_k . $\widehat{\mathbb{D}}_k$ has no discrete torsion (*i.e.* $H^2(\widehat{\mathbb{D}}_k, U(1)) = 0$) and thus there is only one choice of orbifold.

Consider the exact sequence

$$0 \rightarrow \mathbb{Z}_{2k} \rightarrow \widehat{\mathbb{D}}_k \rightarrow \mathbb{Z}_2 \rightarrow 0. \quad (10)$$

This means that $\widehat{\mathbb{D}}_k$ has a normal subgroup \mathbb{Z}_{2k} . We want to show that the quiver for the binary dihedral group may be obtained from the \mathbb{Z}_{2k} quiver in a natural way. In other words, the exact sequence (10) permits us to think about the orbifold $\mathbb{C}^2/\widehat{\mathbb{D}}_k$ in terms of a \mathbb{Z}_2 action on the orbifold $\mathbb{C}^2/\mathbb{Z}_{2k}$ (*i.e.* $\mathbb{C}^2/\widehat{\mathbb{D}}_k \simeq (\mathbb{C}^2/\mathbb{Z}_{2k})/\mathbb{Z}_2$).

The only irreducible representations of the cyclic group \mathbb{Z}_n are n -th roots of unity, $\mathbf{1}_a = \omega_n^a$, where $\omega_n \equiv e^{2\pi i/n}$ and the index $a = 0, 1, \dots, n-1$. The quiver for the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ is the $\widehat{\mathbb{A}}_{n-1}$ quiver in the A - D - E classification.

The binary dihedral group [21] is generated by two elements satisfying

$$e_1 e_2 = e_2 e_1^{-1}, \quad e_1^k = e_2^2, \quad e_1^{2k} = 1. \quad (11)$$

This group, of order $4k$, has four one-dimensional irreducible representations, which we label $\mathbf{1}_j$:

$$\mathbf{1}_0 : (+1, +1), \quad (12)$$

$$\mathbf{1}_1 : (-1, +e^{k\pi i/2}), \quad (13)$$

$$\mathbf{1}_2 : (+1, -1), \quad (14)$$

$$\mathbf{1}_3 : (-1, -e^{k\pi i/2}) \quad (15)$$

and $(k-1)$ two-dimensional irreducible representations, which we label $\mathbf{2}_a$. A particular choice of basis for the $\mathbf{2}$ s is

$$e_1 = \begin{pmatrix} \omega_{2k}^a & 0 \\ 0 & \omega_{2k}^{-a} \end{pmatrix}, \quad e_2 = i^a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (16)$$

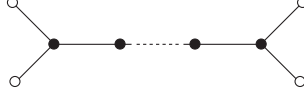


Figure 1: Quiver of $\widehat{\mathbb{D}}_k$ singularity.

with the index $a = 1, \dots, k-1$. All other representations of $\widehat{\mathbb{D}}_k$ are either reducible or are $GL(2, \mathbb{C})$ equivalent to the ones listed here. (In particular, $\mathbf{2}_{k-l}$ and $\mathbf{2}_{k+l}$ are $GL(2, \mathbb{C})$ equivalent.) The quiver for this theory is shown in Figure 1. (Here we use $N = 2$ notation, so a line corresponds to a hypermultiplet (a pair of arrows).)

Now let us reproduce these results using the techniques that we have discussed above. First we must identify the action of \mathbb{Z}_2 on the $\widehat{\mathbb{A}}_{n-1}$ quiver. It is easily seen that the \mathbb{Z}_2 acts by identifying a root of unity with its inverse. Thus, the node for ω_{2k}^{k-a} is identified with

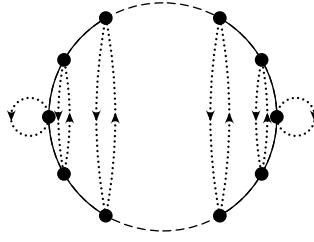


Figure 2: Quiver of $\widehat{\mathbb{A}}_{2k-1}$ singularity with a \mathbb{Z}_2 action.

ω_{2k}^{k+a} for $a = 0, 1, \dots, k$. Note that under this \mathbb{Z}_2 action the nodes corresponding to $\omega_{2k}^0 = 1$ and $\omega_{2k}^k = -1$ map to themselves.

The nodes which are in a 2-orbit of \mathbb{Z}_2 combine to form a 2-node of the new quiver; a node that is in a 1-orbit splits into two 1-nodes. Thus, we get a total of $k-1$ two-dimensional irreducible representations and four one-dimensional irreducible representations. The lines connecting nodes are inherited from the \mathbb{Z}_{2k} quiver: each pair of $\mathbf{1}$ s connects to one of the $\mathbf{2}$ s, while the $\mathbf{2}$ s connect to each other along a line. Thus we reproduce the $\widehat{\mathbb{D}}_k$ quiver, Figure 1.

The same prescription applies for the binary polyhedral groups² $\widehat{\mathbb{E}}_6$ and $\widehat{\mathbb{E}}_7$ since we have the exact sequences [22]

$$0 \rightarrow \widehat{\mathbb{D}}_2 \rightarrow \widehat{\mathbb{E}}_6 \rightarrow \mathbb{Z}_3 \rightarrow 0, \quad (17)$$

$$0 \rightarrow \widehat{\mathbb{E}}_6 \rightarrow \widehat{\mathbb{E}}_7 \rightarrow \mathbb{Z}_2 \rightarrow 0. \quad (18)$$

Diagrammatically, the quivers of these are obtained in Figures 3 and 4, where orbits are denoted by the dotted lines.

²The quiver for $\widehat{\mathbb{E}}_8$ cannot be obtained in this way, as there is no such useful exact sequence.

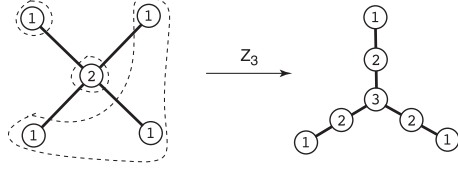


Figure 3: Quiver of $\widehat{\mathbb{E}}_6$ singularity.

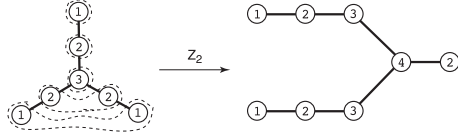


Figure 4: Quiver of $\widehat{\mathbb{E}}_7$ singularity.

3.2 Second Example: $\mathbb{C}^2/\mathbb{Z}_4$

Since \mathbb{Z}_2 is a normal subgroup of \mathbb{Z}_4 , we have the exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0. \quad (19)$$

This is one of the examples where one can choose the lift of \mathbb{Z}_2 to the automorphisms of the quiver to involve gauge transformations. If the left \mathbb{Z}_2 subgroup is generated by e , the right \mathbb{Z}_2 action in the \mathbb{Z}_2 quiver must square to give $\sigma^2 = e$ so that we obtain a \mathbb{Z}_4 action. Thus we see that we must consider here an action of the right \mathbb{Z}_2 which is a representation only up to a gauge transformation.

The quiver diagram of \mathbb{Z}_2 consists of a pair of nodes, $+1$ and -1 connected by a pair of bifundamentals. The group $\mathbb{Z}_4/\mathbb{Z}_2 \simeq \mathbb{Z}_2$ acts trivially on $\widehat{\mathbb{A}}_1$ by sending each node to itself. Rule (1) tells us that the quiver for \mathbb{Z}_4 contains four nodes of rank 1. Rule (2) tells us that each of the nodes that formed from the splitting of the node $+1$ connects to each of the nodes that formed from the splitting of the node -1 via hypermultiplets.

Notice that in the quiver of \mathbb{Z}_2 we have two hypermultiplets, each corresponding to two arrows between the same two representations running in opposite directions. We have four $N = 1$ superfields, two going from node one to node two ϕ_{12}^i , $i = 1, 2$ and two going backwards ϕ_{21}^i . The \mathbb{Z}_2 generated by σ takes $\phi^1 \rightarrow i\phi^1$ and $\phi^2 \rightarrow -i\phi^2$. Notice that σ^2 takes $\phi^i \rightarrow -\phi^i$ for all i , and this can be interpreted as an action by multiplying all the superfields by -1 , which is a gauge transformation on one of the nodes by (-1) .

By this construction, the nodes corresponding to ϕ^1 transform differently than the nodes corresponding to ϕ^2 . In particular the arrows ϕ_{21}^1 and ϕ_{21}^2 will join different representations. Straightening out the crossed lines, we obtain the $\widehat{\mathbb{A}}_3$ quiver. This is illustrated in Figure 5. The orbifold $\mathbb{C}^2/\mathbb{Z}_4$ is an orbifold of an orbifold, $(\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_2$.

Of course, these quivers may be obtained in the standard way without difficulty. The techniques are more powerful, however, when discrete torsion is involved. We shall now



Figure 5: The quiver of the $\widehat{\mathbb{A}}_3$ singularity is obtained as a \mathbb{Z}_2 automorphism of the quiver of the $\widehat{\mathbb{A}}_2$ singularity.

provide a derivation of the quiver rules listed above before analyzing these examples and others, which include discrete torsion, in greater detail.

4 Product Group Orbifolds

In the general case one wants to understand how to add discrete torsion. The purpose of this section is to address this issue. We will begin by considering in particular the orbifold

$$\mathbb{C}^n / (G_1 \times G_2), \quad (20)$$

where the G_i act by linear transformations on the generators of \mathbb{C}^n . In later sections we consider a series of examples, generally of non-Abelian groups with discrete torsion. All of these examples may be thought of as (semi-) direct products, and thus it is important to study the general case in detail.

Since the orbifold group is a direct product, we clearly have an exact sequence

$$0 \rightarrow G_1 \rightarrow G_1 \times G_2 \rightarrow G_2 \rightarrow 0 \quad (21)$$

and so, from the discussion of the previous section, we can consider

$$(\mathbb{C}^n / G_1) / G_2. \quad (22)$$

The general question we want to answer now is the following: what is the quiver diagram for this orbifold given a choice of discrete torsion of the group $G_1 \times G_2$?

To answer the question, first we note that there is a formula for the discrete torsion of $G_1 \times G_2$ [23]:

$$\begin{aligned} H^2(G_1 \times G_2, U(1)) &= H^2(G_1, U(1)) \times H^2(G_2, U(1)) \\ &\times [H^1(G_1, U(1)) \otimes_{\mathbb{Z}} H^1(G_2, U(1))]. \end{aligned} \quad (23)$$

Alternately, by a theorem of Yamazaki [23], we may write

$$H^1(G_1, U(1)) \otimes_{\mathbb{Z}} H^1(G_2, U(1)) = \text{Hom}_{\mathbb{Z}}(G_1/G'_1, G_2/G'_2), \quad (24)$$

where we have $G' \equiv [G, G]$, the commutator subgroup of G .

Thus, discrete torsion has several sources. In light of the structure (22), if $H^2(G_1, U(1)) \neq 0$, this should be taken into account for the quiver corresponding to \mathbb{C}^n / G_1 . We will concentrate on the case where $H^2(G_2, U(1)) = 0$; this is an important simplifying assumption, and

in any case is true for all of the examples that we consider. The remaining source of discrete torsion is the “interaction” term $H^1(G_1, U(1)) \otimes_{\mathbb{Z}} H^1(G_2, U(1))$. The group $H^1(G, U(1))$ is the group of one-dimensional irreducible representations of G , with multiplication given by the tensor product of representations. Generally, we will refer to elements of this group by χ .

4.1 Abelian G_2

The case of Abelian G_2 is particularly simple. If G_2 is the cyclic group \mathbb{Z}_n , we have $H^1(G_2, U(1)) = G_2$, the group of characters of G_2 . In this case we have

$$H^1(G_1, U(1)) \otimes_{\mathbb{Z}} \mathbb{Z}_n \subseteq H^1(G_1, U(1)). \quad (25)$$

A choice of discrete torsion amounts to a choice of subgroup of the characters of G_1 . We now need to consider representations of the group algebra of $G_1 \times \mathbb{Z}_n$ with the choice of discrete torsion.

We will write e as the generator of G_2 . The group algebra $\mathcal{A}(G)$ will be generated by e and the generators of G_1 . Because G is a product group, ge and eg can differ only by a phase, and in fact

$$e \cdot g = g \cdot e\chi(g) \quad (26)$$

for any $g \in G_1$ and $\chi \in H^1(G_1, U(1))$. As a result, we can think of e as acting on the elements (3) of the group algebra

$$e : \sum_i a_i g_i \rightarrow e \cdot \sum_i a_i g_i \cdot e^{-1}, \quad (27)$$

and this is an outer automorphism of $\mathcal{A}(G_1)$. In particular, the action of e leaves the center $\mathcal{Z}\mathcal{A}(G_1)$ invariant. We conclude that it acts on the representations of G_1 by permutations. Because of (26), e takes a representation R to $\chi \otimes R$, which is irreducible and has the same dimension as R .

A first step then in constructing the quiver of $(\mathbb{C}^n/G_1)/G_2$ is to decide on how e permutes the nodes of the \mathbb{C}^n/G_1 quiver. The bottom line is that different choices of discrete torsion correspond to different sets of orbits of nodes.

We will discuss the representations of G_1 in terms of projectors P_i , as in Section 2. In the present case, every element of the center of the algebra can be written as

$$\sum_{i,\ell} a_{i\ell} P_i e^\ell, \quad (28)$$

where $a_{j\ell} = a_{\sigma_e^{-1}(j),\ell}$.

We also have

$$\sum_{\ell} a_{j\ell} (P_j - P_{\sigma_\ell(j)}) e^\ell = 0 \quad (29)$$

for all j . That is, only if e^k acts trivially on the orbit of P_j , can we have $a_{jk} \neq 0$. Thus for each orbit of representations of the group G_1 of order k' , we get n/k' distinct representations, which appear as nodes in the new quiver. Note that a given value of k is a multiple of k' .

To be more specific, we implement the action of e , Eq. (26). Choose a basis $|v_i\rangle_R$ for the vector space where G_1 acts block diagonally. Then $e|v\rangle_R$ transforms under G_1 as $|v\rangle_{\chi \otimes R}$. Consider a matrix representation for e ; we can write this in terms of an invertible matrix U as

$$e|v_i\rangle_R = (U_i^k)_{R, \chi \otimes R} |v_k\rangle_{\chi \otimes R}. \quad (30)$$

If we think of elements of G_1 as block diagonal

$$R(G_1) = \text{diag}(R_0, R_1, R_2, \dots, R_{k'-1}), \quad (31)$$

then e takes the form

$$e \sim \begin{pmatrix} 0 & U_{R_0, R_1} & 0 & \dots & 0 \\ 0 & 0 & U_{R_1, R_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{R_{k'-1}, R_0} & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (32)$$

Here, we have a set of k' d -dimensional representations $R_0, R_1, R_2, \dots, R_{k'-1}$ that are permuted by e (that is, $R_i = \chi \otimes R_{i-1}$). By $GL(k'd, \mathbb{C})$ transformations, we can set the determinants of each submatrix U_{R_i, R_j} to unity, leaving an overall phase α in e . Since $e^n = I$, we must have $\alpha^n = 1$. By acting with e^k , we get an element which belongs to the center of the twisted group algebra. On an irreducible representation this is proportional to the identity. Thus $e^k = \text{diag}((U_{R_0, R_1} \dots U_{R_{k'-1}, R_0})^{k/k'} \alpha^k)$. Without loss of generality, we can take $(U_{R_0, R_1} \dots U_{R_{k'-1}, R_0})^{k/k'} = I$, and thus $e^k = \alpha^k I$.

Since α is an n -th root of unity it might appear that we get n distinct representations this way. However, there is an $SL(k'd)$ transformation that takes $\alpha \rightarrow \alpha \omega_{k'}$, where $\omega_{k'}^{k'} = 1$. Thus, we reproduce the result that there are n/k' distinct irreducible representations of dimension dk' of the group for each of these orbits.

Notice that if we sum the square of the dimensions of the representations over all orbits we get

$$\sum_{orb} \frac{n}{k'} (k'd)^2 = \sum_{orb} n d^2 k' = n \sum_{orb} d^2 k'. \quad (33)$$

Each orbit is made of exactly k' distinct irreducible (projective) representations of G_1 , so the sum over orbits covers the sum over irreducible representations of G_1 . Thus

$$\sum_{orb} \frac{n}{k'} (k'd)^2 = n \sum_R \dim(R)^2 = n |G_1| = |G_1 \times \mathbb{Z}_n|. \quad (34)$$

This equality can only hold if the sum in the left is over all the possible distinct irreducible representations of the group $|G_1 \times \mathbb{Z}_n|$ with a given cocycle. Thus we have obtained all of the nodes of the quiver in this way.

4.2 Non-Abelian G_2

If G_2 is non-Abelian it is not true in general that $H^1(G_2, U(1))$ is equivalent to G_2 ; there is an exact sequence of groups

$$0 \rightarrow \tilde{G}_2 \rightarrow G_2 \rightarrow H^1(G_2, U(1)) \rightarrow 0 \quad (35)$$

with \tilde{G}_2 the kernel of the group map. In such a case, an orbifold \mathcal{M}/G_2 may be understood as $(\mathcal{M}/\tilde{G}_2)/H^1(G_2, U(1))$; this reduces the non-Abelian quotient to an Abelian one, provided we understand how the discrete torsion behaves for a sequence of groups as above. In general this particular problem is complicated,³ but we can apply the same ideas as before to build representations.

5 Examples: Discrete Subgroups of $SU(3)$

Discrete torsion in orbifolds of the Abelian product groups has been extensively studied [25, 11, 12, 26, 13, 14, 15], so we utilize the tools we have developed for constructing quivers to analyze several non-Abelian (semi-) direct product groups. The dedicated reader may find it instructive to apply our methods to the $\mathbb{Z}_m \times \mathbb{Z}_n$ case. Here, we will focus on some of the non-Abelian discrete subgroups of $SU(3)$. These have been listed recently in Ref. [20], where the discrete torsion was computed. We should note here that there is one important subtlety if we want to consider subgroups of $SU(3)$: namely, the group that was written in Ref. [20] as $G \times \mathbb{Z}_n$ (where $G = \widehat{\mathbb{D}}_k, \widehat{\mathbb{E}}_k$) is in actuality $(G \times \mathbb{Z}_{2n})/\mathbb{Z}_2$. This point will be explained in the following; the results for discrete torsion are modified accordingly.

5.1 $\mathbb{C}^3/[(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2]$

Since $H^2(\widehat{\mathbb{D}}_k, U(1))$ and $H^2(\mathbb{Z}_n, U(1))$ are both trivial, the direct product group $\widehat{\mathbb{D}}_k \times \mathbb{Z}_n$ has discrete torsion given by $\text{Hom}_{\mathbb{Z}}(\widehat{\mathbb{D}}_k/\widehat{\mathbb{D}}'_k, \mathbb{Z}_n)$. For even k , $\widehat{\mathbb{D}}_k/\widehat{\mathbb{D}}'_k$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and for odd k it is \mathbb{Z}_4 . Thus [20],

$$H^2(\widehat{\mathbb{D}}_k \times \mathbb{Z}_n, U(1)) = \begin{cases} \mathbb{Z}_4 & k \text{ odd}, n = 0 \bmod 4, \\ \mathbb{Z}_2 & k \text{ odd}, n = 2 \bmod 4, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & k \text{ even}, n \text{ even}, \\ 1 & n \text{ odd}. \end{cases} \quad (36)$$

However, $\widehat{\mathbb{D}}_k \times \mathbb{Z}_n \subset SU(2) \times U(1)$ is not a subgroup of $SU(3)$ for even n . Rather, we should consider the group $(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2 \subset (SU(2) \times U(1))/\mathbb{Z}_2$.

To unravel this technical point, let us consider the group action of the generators of the product group $\widehat{\mathbb{D}}_k \times \mathbb{Z}_n$. Let e_0 be the generator of \mathbb{Z}_n and e_1, e_2 be the generators of $\widehat{\mathbb{D}}_k$,

³The calculation of the group cohomology groups can be carried out in their classifying spaces. Each sequence of groups induces a fibration and one can compute some approximation to the cohomology via a spectral sequence [24]. The difficult point is to know if the spectral sequence approximation and the cohomology actually agree.

whose group algebra is given in Eq. (11). Then the generators act on the coordinates of \mathbb{C}^3 as follows.

$$\begin{aligned} e_0 : & \quad (z_1, z_2, z_3) \rightarrow (\omega_n z_1, \omega_n z_2, \omega_n^{-2} z_3), \\ e_1 : & \quad (z_1, z_2, z_3) \rightarrow (\omega_{2k} z_1, \omega_{2k}^{-1} z_2, z_3), \\ e_2 : & \quad (z_1, z_2, z_3) \rightarrow (iz_2, iz_1, z_3). \end{aligned} \tag{37}$$

When n is even, $e_0^{n/2}$ and e_1^k have the same action on the space. The group action is not faithful. (This was pointed out in the context of brane box models in Refs. [27, 28].) Naïvely applying the quiver rules with discrete torsion given by Eq. (36) yields incorrect results.

The group $G = (\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ mods out by the ambiguity in the group action. To determine the discrete torsion that G admits, we examine the twisted group algebra explicitly. We have the relations

$$\begin{aligned} e_0^n = e_1^k = e_2^2, \quad e_0^{2n} = 1, \\ e_0 e_1 = \theta e_1 e_0, \quad e_0 e_2 = \eta e_2 e_0, \quad e_1 e_2 = e_2 e_1^{-1}, \end{aligned} \tag{38}$$

where θ and η are phases inherent to the projective representations of the algebra [11, 12]. They encode the characters of $(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2$. The above relations imply that

$$\theta^n = \theta^k = \theta^2 = \eta^n = \eta^2 = 1. \tag{39}$$

Thus,

$$(\theta, \eta) = \begin{cases} (\pm 1, \pm 1) & k \text{ even, } n \text{ even,} \\ (+1, \pm 1) & k \text{ odd, } n \text{ even,} \\ (+1, +1) & n \text{ odd,} \end{cases} \tag{40}$$

which means that

$$H^2((\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2, U(1)) = \begin{cases} \mathbb{Z}_2 & k \text{ odd, } n \text{ even,} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & k \text{ even, } n \text{ even,} \\ 1 & n \text{ odd.} \end{cases} \tag{41}$$

This result is consistent with the long exact sequence in cohomology obtained from

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n} \rightarrow (\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2 \rightarrow 0 \tag{42}$$

using theorems of Hochschild and Serre and Iwahori and Matsumoto, which may be found in Ref. [23]. We note that Eq. (41) differs from Eq. (36) in a crucial way. We now conclude that whenever k is odd and n is even the discrete torsion is \mathbb{Z}_2 . The $n = 0 \bmod 4$ and $n = 2 \bmod 4$ cases are not distinguished.

5.1.1 The $N = 1$ Quiver

To construct the nodes of the quiver, we can consider the sequence

$$0 \rightarrow \widehat{\mathbb{D}}_k \rightarrow (\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2 \rightarrow \mathbb{Z}_n \rightarrow 0. \quad (43)$$

The nodes of the quiver can thus be constructed by considering a \mathbb{Z}_n action on the $\widehat{\mathbb{D}}_k$ quiver. Without discrete torsion, all orbits are one-dimensional, and thus we obtain a stack of n copies of the $\widehat{\mathbb{D}}_k$ quiver, with arrows to be determined.

We can see this structure directly. Without discrete torsion, we have the relations

$$\begin{aligned} e_0^n &= e_1^k = e_2^2, & e_0^{2n} &= 1, \\ e_0 e_1 &= e_1 e_0, & e_0 e_2 &= e_2 e_0, & e_1 e_2 &= e_2 e_1^{-1}. \end{aligned} \quad (44)$$

This algebra possesses $4n$ one-dimensional representations:

$$(e_0, e_1, e_2) = \begin{cases} (\omega_{2n}^{2a}, +1, \pm 1), \\ (\omega_{2n}^{2a}, -1, \pm 1) & k \text{ even}, \\ (\omega_{2n}^{2a+1}, -1, \pm i) & k \text{ odd}, \end{cases} \quad (45)$$

where $a = 0, 1, \dots, n-1$. The $n(k-1)$ two-dimensional irreducible representations can be written as

$$e_0 = \begin{pmatrix} \omega_{2n}^b & 0 \\ 0 & \omega_{2n}^b \end{pmatrix}, \quad e_1 = \begin{pmatrix} \omega_{2k}^c & 0 \\ 0 & \omega_{2k}^{-c} \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i^c \\ i^c & 0 \end{pmatrix}, \quad (46)$$

where $b = 0, 1, \dots, 2n-1$, $c = 1, 2, \dots, k-1$, and $b = c \bmod 2$. The nodes of the $(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ quiver are arrayed as $2n$ copies of half a $\widehat{\mathbb{D}}_k$ quiver. We make the observation that when e_0 is an even power of ω_{2n} , e_1 and e_2 are representations of \mathbb{D}_k , whereas when the power of ω_{2n} is odd, e_1 and e_2 are representations of \mathbb{D}_k with \mathbb{Z}_2 discrete torsion turned on.

To determine how the arrows are drawn, we note the action of the generators on \mathbb{C}^3 .

$$\begin{aligned} e_0 : & (z_1, z_2, z_3) \rightarrow (\omega_{2n} z_1, \omega_{2n} z_2, \omega_{2n}^{-2} z_3), \\ e_1 : & (z_1, z_2, z_3) \rightarrow (\omega_{2k} z_1, \omega_{2k}^{-1} z_2, z_3), \\ e_2 : & (z_1, z_2, z_3) \rightarrow (i z_2, i z_1, z_3). \end{aligned} \quad (47)$$

Thus, the orbifold group acts as $(\omega_{2n}, \mathbf{2}_1)$ on (ϕ_1, ϕ_2) and as $(\omega_{2n}^{-2}, \mathbf{1}_0)$ on ϕ_3 . To fill in the lines of the quiver, we consider the decompositions $R \otimes R_i = \oplus R_j$, drawing a (chiral) line from R_i to R_j .

For (ϕ_1, ϕ_2) we have

$$(\omega_{2n}, \mathbf{2}_1) \otimes (\omega_{2n}^\ell, \mathbf{1}_j) = (\omega_{2n}^{\ell+1}, \mathbf{2}_{x(j)}), \quad (48)$$

$$(\omega_{2n}, \mathbf{2}_1) \otimes (\omega_{2n}^\ell, \mathbf{2}_a) = (\omega_{2n}^{\ell+1}, \mathbf{4}_a), \quad (49)$$

where $x(0) = x(2) = 1$ and $x(1) = x(3) = k-1$. The representations $\mathbf{4}_a$ are reducible as follows: $\mathbf{4}_1 = \mathbf{2}_2 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2$, $\mathbf{4}_{k-1} = \mathbf{2}_{k-2} \oplus \mathbf{1}_1 \oplus \mathbf{1}_3$ and $\mathbf{4}_a = \mathbf{2}_{a-1} \oplus \mathbf{2}_{a+1}$ for $a \neq 1, k-1$.

For ϕ_3 , we have

$$(\omega_{2n}^{-2}, \mathbf{1}_0) \otimes (\omega_{2n}^\ell, \mathbf{1}_j) = (\omega_{2n}^{\ell-2}, \mathbf{1}_j), \quad (50)$$

$$(\omega_{2n}^{-2}, \mathbf{1}_0) \otimes (\omega_{2n}^\ell, \mathbf{2}_a) = (\omega_{2n}^{\ell-2}, \mathbf{2}_a). \quad (51)$$

As anticipated, the resulting quiver may be thought of as a stack of quivers of the form of Figure 1 with n levels. The fields ϕ_1, ϕ_2, ϕ_3 cause interconnections between the levels. This is rather complicated to draw in general, and we will show only the interconnections between two levels of the stack. The ϕ_1 and ϕ_2 arrows from $\mathbf{1}_2$ and $\mathbf{1}_3$ for odd k are slightly different than what is shown in Figure 6 in that they connect the two levels of the stack.

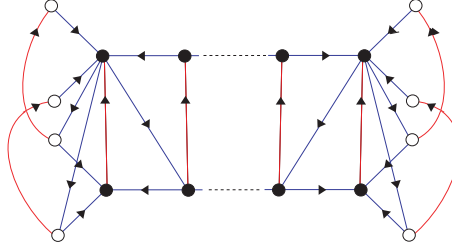


Figure 6: A part of the $(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ quiver.

5.1.2 $H^2((\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$

When n is even and k is odd, the $(\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ algebra admits a \mathbb{Z}_2 discrete torsion. This is implemented as a modification of the algebra in Eq. (44). Now,

$$e_0 e_2 = -e_2 e_0. \quad (52)$$

The only irreducible representations of this algebra are two-dimensional. Up to conjugation by elements of $GL(2, \mathbb{C})$, we have

$$e_0 = \begin{pmatrix} \omega_{2n}^a & 0 \\ 0 & -\omega_{2n}^a \end{pmatrix}, \quad e_1 = \begin{pmatrix} \omega_{2k}^b & 0 \\ 0 & \omega_{2k}^{-b} \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i^b \\ i^b & 0 \end{pmatrix}, \quad (53)$$

where $a = 0, 1, \dots, n-1$, $b = 0, 1, \dots, 2k-1$, and $a = b \bmod 2$. Thus, there are nk **2**s. The representations for which $b = 0, k$ correspond to the **2**s built from combining **1**s. All the representations found in the original quiver are accounted for:

$$4n \cdot 1^2 + n(k-1) \cdot 2^2 = nk \cdot 2^2. \quad (54)$$

This is precisely what the quiver rules listed in Section 3 tell us. Consider as an example the quiver of $(\widehat{\mathbb{D}}_3 \times \mathbb{Z}_4)/\mathbb{Z}_2$. Without discrete torsion, the quiver consists of two copies of the $\widehat{\mathbb{D}}_3$ quiver with interconnections. With discrete torsion, the \mathbb{Z}_2 acts to combine the **1**₀ and **1**₂ into a **2**. The **1**₁ and **1**₃ representations combine to give a second **2**. The two **2**s in the

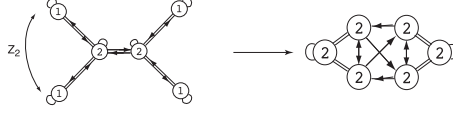


Figure 7: $(\widehat{\mathbb{D}}_3 \times \mathbb{Z}_4)/\mathbb{Z}_2$ with \mathbb{Z}_2 torsion.

center of the quiver split into four **2**s since they are acted upon trivially by \mathbb{Z}_2 . Thus, the final quiver consists of six nodes, all of which are **2**s.

The ϕ_3 lines are adjoints for the **2**s that were constructed from the one-dimensional irreducible representations of $\widehat{\mathbb{D}}_3$. In the $\widehat{\mathbb{D}}_3$ quiver, a hypermultiplet connected a **1** to its neighboring **2**, so now, a hypermultiplet connects the **2** built from the **1**s to each of the daughters of the neighbor which split. The lines between the **2**s that formed through splitting are chiral in the new quiver, which mimics the structure of the links in the $(\widehat{\mathbb{D}}_3 \times \mathbb{Z}_4)/\mathbb{Z}_2$ model without discrete torsion.

The interconnections become increasingly complicated as n and k increase. For example, the quiver with discrete torsion for a $(\widehat{\mathbb{D}}_3 \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ model consists of $n/2$ interconnected copies of the quiver shown in Figure 7. These theories are generically chiral.

5.1.3 $H^2((\widehat{\mathbb{D}}_k \times \mathbb{Z}_{2n})/\mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$

We have $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete torsion if both n and k are even. When considering the algebra, the relations in Eq. (44) are modified as follows:

$$e_0 e_1 = \pm e_1 e_0, \quad e_0 e_2 = \pm e_2 e_0. \quad (55)$$

The four sign choices correspond to the four **1**s of $\widehat{\mathbb{D}}_k$. A non-trivial automorphism of the quiver maps **1**₀ to one of the other one-dimensional irreducible representations, which then maps back to **1**₀.

If **1**₀ and **1**₂ lie within an orbit, then the torsion acts analogously to the even n , odd k case discussed above. We choose signs $(+, -)$ in Eq. (55). The only irreducible representations consistent with this choice are **2**s, and, up to $GL(2, \mathbb{C})$ equivalence, these have the explicit realization given in Eq. (53). Once again, $b = 0, k$ correspond to two-dimensional representations that form when one-dimensional representations combine while the other values of b correspond to the splitting of the two-dimensional nodes in the torsion-free quiver. The quiver for $(\widehat{\mathbb{D}}_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$ with this choice of torsion is given in Figure 8. In the generic case where $n > 2$, we obtain $n/2$ interconnected copies of this quiver.

If **1**₀ lies in an orbit with either **1**₁ or **1**₃, there are four-dimensional irreducible representations of the algebra as well. The key observation here is that it is only when the discrete torsion acts to produce orbits which cross the vertical axis of the quiver that we get higher dimensional irreducible representations from orbits that map pairs of **2**s to each other. Such a result was impossible for odd k because the \mathbb{Z}_2 discrete torsion is incompatible with the

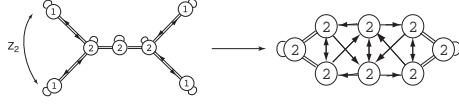


Figure 8: $(\widehat{\mathbb{D}}_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete torsion: $\mathbf{1}_0 \leftrightarrow \mathbf{1}_2$.

choice of characters corresponding to $\mathbf{1}_1$ and $\mathbf{1}_3$. That is to say, we could not introduce factors of $\pm i$ into the twisted algebra.

An explicit realization of the $\mathbf{1}_0 \leftrightarrow \mathbf{1}_3$ orbit, which corresponds to the sign choice $(-, -)$ in Eq. (55), is provided below. The $2n$ $\mathbf{2}$ s are

$$e_0 = \omega_n^a \sigma^1, \quad e_1 = \sigma^3, \quad e_2 = \pm \sigma^3, \quad (56)$$

where σ^i are the spin-1/2 Pauli matrices and $a = 0, 1, \dots, n-1$. The $\mathbf{4}$ s are built by using direct sums of the representations of $\widehat{\mathbb{D}}_k$ for e_1 and e_2 and then solving for e_0 using the relations of the twisted algebra. An explicit realization is

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & \alpha \omega_n^a & 0 & 0 \\ \omega_n^a & 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} \omega_{2k}^b & 0 & 0 & 0 \\ 0 & \omega_{2k}^{-b} & 0 & 0 \\ 0 & 0 & \omega_{2k}^{k-b} & 0 \\ 0 & 0 & 0 & \omega_{2k}^{-(k-b)} \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i^b & 0 & 0 \\ i^b & 0 & 0 & 0 \\ 0 & 0 & 0 & i^{k-b} \\ 0 & 0 & i^{k-b} & 0 \end{pmatrix}, \quad (57)$$

where $a = 0, 1, \dots, n-1$, $b = 1, 2, \dots, (k-2)/2$, and $\alpha \equiv (-1)^{k/2-b+1}$. The b compatible with a given a are those for which the relation

$$(-1)^a = (-1)^b \alpha^{n/2} \quad (58)$$

is true. In the end, there are $n(k-2)/4$ such $\mathbf{4}$ s. The quiver for the $(\widehat{\mathbb{D}}_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$ model with this choice of torsion is given in Figure 9. Unlike the quiver obtained from the $\mathbf{1}_2$ choice

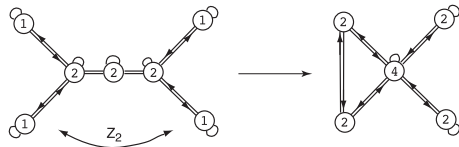


Figure 9: $(\widehat{\mathbb{D}}_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete torsion: $\mathbf{1}_0 \leftrightarrow \mathbf{1}_3$.

of characters, this theory is non-chiral.

5.2 $\mathbb{C}^3/[(\widehat{\mathbb{E}}_6 \times \mathbb{Z}_{2n})/\mathbb{Z}_2]$, $\mathbb{C}^3/[(\widehat{\mathbb{E}}_7 \times \mathbb{Z}_{2n})/\mathbb{Z}_2]$

We now explore a network of exact sequences that correlate the ordinary tetrahedral and octahedral groups to their double covers. Algebraic details regarding the representation theory of polyhedral groups are relegated to Appendix E. We then explore introducing discrete torsion into orbifolds of product groups involving $\widehat{\mathbb{E}}_6$ and $\widehat{\mathbb{E}}_7$.

5.2.1 $\mathbb{C}^2/\widehat{\mathbb{E}}_6$ Revisited

In Section 3, we observed that the quiver for $\widehat{\mathbb{E}}_6$ arises from a \mathbb{Z}_3 action on the quiver for $\widehat{\mathbb{D}}_2$ and that the quiver for $\widehat{\mathbb{E}}_7$ arises from a \mathbb{Z}_2 action on the quiver for $\widehat{\mathbb{E}}_6$ because of the exact sequences Eqs. (17–18). In fact, the structure is much richer. In the following discussion, we specialize to the case of $\widehat{\mathbb{E}}_6$. A similar story holds for $\widehat{\mathbb{E}}_7$.

The binary tetrahedral group is at the center of the following web of exact sequences:

$$\begin{array}{ccccc} \mathbb{Z}_2 & & \mathbb{Z}_2 & & \\ \downarrow & & \downarrow & & \\ \widehat{\mathbb{D}}_2 & \rightarrow & \widehat{\mathbb{E}}_6 & \rightarrow & \mathbb{Z}_3 \\ \downarrow & & \downarrow & & \\ \mathbb{D}_2 & \rightarrow & \mathbb{E}_6 & \rightarrow & \mathbb{Z}_3 \end{array} \quad (59)$$

The middle horizontal line of this web is the construction of $\widehat{\mathbb{E}}_6$ in Section 3. The bottom line is the analogous construction of \mathbb{E}_6 from \mathbb{D}_2 . The ordinary dihedral group \mathbb{D}_2 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which admits \mathbb{Z}_2 discrete torsion [11, 12, 13]. The ordinary tetrahedral group has the same discrete torsion: $H^2(\mathbb{E}_6, U(1)) = \mathbb{Z}_2$ [29]. Figure 10 shows the non-chiral \mathbb{E}_6 quivers, both without and with \mathbb{Z}_2 torsion, that are built from imposing a \mathbb{Z}_3 action on the \mathbb{D}_2 quiver without and with \mathbb{Z}_2 torsion.



Figure 10: Building the \mathbb{E}_6 quiver: (a) without discrete torsion, (b) with discrete torsion.

The vertical strands of the web indicate another way in which the $\widehat{\mathbb{E}}_6$ and $\widehat{\mathbb{D}}_2$ quivers may be conceived. Namely,

$$\begin{aligned} \mathbb{C}^2/\widehat{\mathbb{E}}_6 &\simeq (\mathbb{C}^2/\widehat{\mathbb{D}}_2)/\mathbb{Z}_3 \\ &\simeq (\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{E}_6; \end{aligned} \quad (60)$$

$$\begin{aligned} \mathbb{C}^2/\widehat{\mathbb{D}}_2 &\simeq (\mathbb{C}^2/\mathbb{Z}_4)/\mathbb{Z}_2 \\ &\simeq (\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{D}_2. \end{aligned} \quad (61)$$

In the language of Section 4, these are examples where G_2 is non-Abelian, and hence, $H^1(G_2, U(1))$ is different from G_2 . The orbifold $(\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{E}_6$ can be thought of as an \mathbb{E}_6 action on the \mathbb{Z}_2 lattice. Each node is identified with one of the \mathbb{E}_6 quivers in Figure 10, and the arrows are such that we recover the $\widehat{\mathbb{E}}_6$ quiver from this construction. In the case of Eq. (61), this prescription is precisely the reverse of the construction of the ordinary dihedral quivers discussed in Ref. [20].

5.2.2 $H^2((\widehat{\mathbb{E}}_6 \times \mathbb{Z}_{2n})/\mathbb{Z}_2, U(1)) = \mathbb{Z}_3$

Because both $\widehat{\mathbb{E}}_6$ and \mathbb{Z}_{2n} contain an element which acts as

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (62)$$

on the coordinates (z_1, z_2, z_3) of \mathbb{C}^3 , we must quotient out by \mathbb{Z}_2 just as we did when considering the direct product of the binary dihedral and cyclic groups. Examining the twisted algebra of $(\widehat{\mathbb{E}}_6 \times \mathbb{Z}_{2n})/\mathbb{Z}_2$, we find that there is a \mathbb{Z}_3 discrete torsion when $n = 0 \bmod 3$. The relations

$$e_0 e_1 = \omega_3^a e_1 e_0, \quad e_0 e_2 = \omega_3^{-a} e_2 e_0, \quad (63)$$

encode the phases, where e_0 is the generator of \mathbb{Z}_{2n} and e_1 and e_2 are generators of $\widehat{\mathbb{E}}_6$, whose algebra is given in Eq. (74). The generators of the product group act on the coordinates of \mathbb{C}^3 by

$$\begin{aligned} e_0 : & \quad (z_1, z_2, z_3) \rightarrow (\omega_{2n} z_1, \omega_{2n} z_2, \omega_n^{-2} z_3), \\ e_1 : & \quad (z_1, z_2, z_3) \rightarrow (\mathbf{2}_0(e_1)(z_1, z_2), z_3), \\ e_2 : & \quad (z_1, z_2, z_3) \rightarrow (\mathbf{2}_0(e_2)(z_1, z_2), z_3), \end{aligned} \quad (64)$$

where $\mathbf{2}_0(e_i)$ are the generators of the defining representation of $\widehat{\mathbb{E}}_6$ as given in Eqs. (76–77). The superfields (ϕ_1, ϕ_2) act as $(\omega_{2n}, \mathbf{2}_0)$ while ϕ_3 acts as $(\omega_{2n}^{-2}, \mathbf{1}_0)$. The construction of the $N = 1$ quiver without discrete torsion follows the discussion in Section 5.1.1. The $3n$ one-dimensional and n three-dimensional representations are characterized by even powers of ω_{2n} and the $3n$ two-dimensional representations by odd powers. In the two cases, the matrix realizations of e_1 and e_2 are precisely the irreducible representations of the ordinary tetrahedral group \mathbb{E}_6 with trivial and non-trivial \mathbb{Z}_2 torsion, respectively. The quiver consists of an interconnected stack of n copies of the $\widehat{\mathbb{E}}_6$ quiver.

Let us briefly apply the same techniques that we have previously employed to study the quiver of $(\widehat{\mathbb{E}}_6 \times \mathbb{Z}_6)/\mathbb{Z}_2$ with the \mathbb{Z}_3 discrete torsion turned on. The torsion acts to produce three orbits, one cycling the **1**s, another cycling the **2**s, and the third leaving the **3** fixed. We therefore expect four **3**s and a **6** in the final quiver. Hypermultiplets run between the **6** and each of the **3**s, and an adjoint sits on each node formed by joining. Chiral lines connect the nodes formed by splitting. This is illustrated in Figure 11. The $(\widehat{\mathbb{E}}_6 \times \mathbb{Z}_{6n})/\mathbb{Z}_2$ model with discrete torsion contains n interconnected copies of this quiver.

The story for $(\widehat{\mathbb{E}}_7 \times \mathbb{Z}_{2n})/\mathbb{Z}_2$ unfolds along similar lines. Since $H^2(\widehat{\mathbb{E}}_7 \times \mathbb{Z}_{2n})/\mathbb{Z}_2, U(1) = \mathbb{Z}_2$, there is discrete torsion here as well.

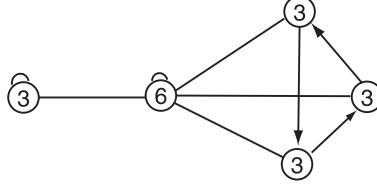


Figure 11: $(\widehat{\mathbb{E}}_6 \times \mathbb{Z}_6)/\mathbb{Z}_2 \sim \widehat{\mathbb{E}}_6 \times \mathbb{Z}_3$ with \mathbb{Z}_3 discrete torsion.

5.3 $\mathbb{C}^3/\Delta_{3n^2,6n^2}$

The group Δ_{3n^2} is a discrete subgroup of $SU(3)$ given by the exact sequence

$$0 \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \Delta_{3n^2} \rightarrow \mathbb{Z}_3 \rightarrow 0. \quad (65)$$

The \mathbb{Z}_3 acts on the coordinates of \mathbb{C}^3 by permutation and the \mathbb{Z}_n acts by phases:

$$\begin{aligned} e_0 : & \quad (z_1, z_2, z_3) \rightarrow (z_3, z_1, z_2), \\ e_1 : & \quad (z_1, z_2, z_3) \rightarrow (\omega_n z_1, \omega_n^{-1} z_2, z_3), \\ e_2 : & \quad (z_1, z_2, z_3) \rightarrow (z_1, \omega_n z_2, \omega_n^{-1} z_3). \end{aligned} \quad (66)$$

From this action, we obtain the relations

$$e_0^3 = e_1^n = e_2^n = e_1 e_2 e_1^{-1} e_2^{-1} = 1, \quad (67)$$

$$e_0 e_1 e_0^{-1} = e_2, \quad e_0 e_2 e_0^{-1} = e_1^{-1} e_2^{-1}. \quad (68)$$

The discrete torsion of Δ_{3n^2} [20] is

$$H^2(\Delta_{3n^2}, U(1)) = \begin{cases} \mathbb{Z}_n \times \mathbb{Z}_3 & n = 0 \bmod 3, \\ \mathbb{Z}_n & \text{otherwise.} \end{cases} \quad (69)$$

The exact sequence (65) is a case where $H^2(G_1, U(1))$ is non-trivial. Thus the discrete torsion arises from two sources: we may embed a phase in the algebra of $\mathbb{Z}_n \times \mathbb{Z}_n$, and, when $n = 0 \bmod 3$, there is a second \mathbb{Z}_3 phase which may be identified with the ‘interaction’ term of Eq. (68). Note that Δ_{3n^2} is *not* a direct product, and as such we should consider where discrete torsion phases are allowed to enter. The general rule is that they can be associated with any element of N which commutes with G/N , as $eg = \chi(g)ge$. In the present case, this is $g = e_1^{n/3} e_2^{2n/3}$, and χ is a \mathbb{Z}_3 phase.

We note the special cases $\Delta_{3,1^2} \simeq \mathbb{Z}_3$ and $\Delta_{3,2^2} \simeq \mathbb{E}_6$ [30]. Other examples may be constructed in a similar fashion.

When $n \neq 0 \bmod 3$, the Δ_{3n^2} quiver is constructed from the $\mathbb{Z}_n \times \mathbb{Z}_n$ lattice. The \mathbb{Z}_3 acts to produce $(n^2 - 1)/3$ orbits among three nodes plus one orbit that leaves a single node fixed. Thus, in the absence of discrete torsion, the quiver for $\Delta_{3,4^2}$, for example, has five **3**s and three **1**s. The theory is chiral. If we add \mathbb{Z}_n discrete torsion, the quiver is reduced to

a $\mathbb{Z}_k \times \mathbb{Z}_k$ lattice, k depending on the particular torsion element. Taking the maximal \mathbb{Z}_4 torsion on $\Delta_{3,4^2}$ for example gives us an $\widehat{\mathbb{A}}_2(4)$ quiver while $\mathbb{Z}_2 \subset \mathbb{Z}_4$ gives an $\mathbb{E}_6(2)$ quiver.⁴

If $n = 0 \bmod 3$, the \mathbb{Z}_3 acts on the $\mathbb{Z}_n \times \mathbb{Z}_n$ lattice to give nine one-dimensional and $n^2/3 - 1$ three-dimensional irreducible representations [31]. If we consider the maximal discrete torsion, then we get the same result as before, namely, the quiver diagram of the \mathbb{Z}_3 orbifold where the nodes are now of rank n (*i.e.* $\widehat{\mathbb{A}}_2(n)$). We are interested in studying how the second \mathbb{Z}_3 torsion acts on the $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver.

Let us now specialize to the case $n = 3$. Without discrete torsion, there are three fixed nodes and two orbits of order three under the action of e_0 on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ lattice. Thus, we obtain the quiver of Δ_{27} in Figure 12.

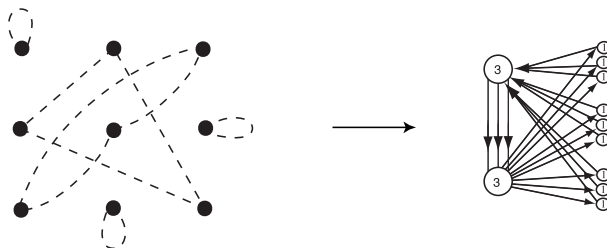


Figure 12: A \mathbb{Z}_3 automorphism of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ lattice gives the quiver of Δ_{27} (no discrete torsion).

If we admit the discrete torsion of the second \mathbb{Z}_3 , then we change the \mathbb{Z}_3 orbits. We find that this produces three 3-node orbits, which consist of one of the 1-node orbits together with a node from each of the 3-node orbits from before. Thus, the resulting quiver is also $\widehat{\mathbb{A}}_2(3)$. We get the *same* quiver diagram for any choice of discrete torsion phases. However, the superpotentials of two such quivers must differ because string theory is sensitive to the choices we have made.

The group Δ_{6n^2} , which is a subgroup of $SU(3)$ for even n [32], fits into the exact sequence

$$0 \rightarrow \Delta_{3n^2} \rightarrow \Delta_{6n^2} \rightarrow \mathbb{Z}_2 \rightarrow 0. \quad (70)$$

Hence, its quiver may be constructed from the Δ_{3n^2} quiver using the techniques we have discussed. The authors of Ref. [20] calculate that $H^2(\Delta_{6n^2}, U(1)) = \mathbb{Z}_2$.

6 Dualities

We have seen that different orbifolds may have the same quiver diagram. Thus, there are different orbifold points in the moduli space of couplings of the corresponding gauge theory. Then from the AdS/CFT correspondence [33, 34, 35], we obtain the result that different orbifolds are on the same moduli space.

A non-trivial example is provided by the $\widehat{\mathbb{A}}_2$ quiver, which is shown in Figure 13. This is

⁴Here the notation $G(n)$ refers to the quiver obtained from the quiver of G by multiplying the rank of each node by n (Morita equivalence).



Figure 13: The $\widehat{\mathbb{A}}_2$ quiver.

the quiver for the $N = 2$ model with superpotential

$$W_1 = \text{tr}[\phi_1, \phi_2]\phi_3. \quad (71)$$

Here, ϕ_1 is a $3m \times 3m$ matrix representing the adjoint fields, while ϕ_2 and ϕ_3 contain the bifundamental fields.

Alternatively, we may obtain the same quiver from a $\mathbb{Z}_{3n} \times \mathbb{Z}_n$ orbifold. Here we choose maximal discrete torsion to obtain the $U(m)^3$ model. The superpotential here may be written

$$W_1 = \text{tr}[\phi_1, \phi_2]\phi_3 + \left(\frac{1-q}{1+q} \right) \text{tr}\{\phi_1, \phi_2\}\phi_3, \quad (72)$$

where q is a $3n$ -th root of unity.

From the above discussions, we know that Δ_{3n^2} may be thought of in terms of the exact sequence $0 \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \Delta_{3n^2} \rightarrow \mathbb{Z}_3 \rightarrow 0$. As a result, the theory is constructed by a \mathbb{Z}_3 projection of the $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold. The \mathbb{Z}_3 permutes the three fields $\phi_{1,2,3}$. This gives rise again to the $\widehat{\mathbb{A}}_2$ quiver. We may define fields $\chi_{1,2,3}$ which transform by rephasing; in terms of these fields, which have the same interpretation as the fields of the $\mathbb{Z}_{3n} \times \mathbb{Z}_n$ orbifold, we find a superpotential

$$W_2 = \text{tr}[\chi_1, \chi_2]\chi_3 + \left(\frac{1-q^3}{1+q^3} \right) \left[\frac{i}{\sqrt{3}} \text{tr}\{\chi_1, \chi_2\}\chi_3 + \frac{1}{3} \text{tr}(\chi_1^3 + \chi_2^3 + \chi_3^3) \right]. \quad (73)$$

The superpotential (72) is a marginal perturbation of (71), reminiscent of Refs. [14, 15]. The superpotential (73) is more interesting: it contains *both* of the marginal perturbations of Ref. [36].

7 Conclusion

In this paper we have presented a technique which permits calculations of quiver diagrams for certain orbifold singularities very efficiently. Presumably these techniques can also be applied to orientifolds with discrete torsion [37, 38] and to other singularities such as orbifolds of conifolds [39].

Quiver diagrams inherit the quantum symmetry of an orbifold, so one can also orbifold a quiver by a subgroup of its quantum symmetry. Although we did not show this explicitly in the paper, our techniques applied in this case give the correct quiver diagram of the (partially) unorbifolded orbifold.

With some other results in the literature [30, 40, 27] these techniques should provide a complementary set of tools to study D-brane field theories. We note also that after this work had been completed, we became aware of the paper [41] which addresses similar issues using different techniques.

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8 Appendix \mathbb{E}

The binary tetrahedral group $\widehat{\mathbb{E}}_6$ is defined by the relations [21]

$$e_1^3 = e_2^3 = (e_2 e_1)^2. \quad (74)$$

The one-dimensional irreducible representations are

$$\mathbf{1}_a : (e_1, e_2) = (\omega_3^a, \omega_3^{-a}) \quad (75)$$

for $a = 0, 1, 2$. In terms of the quaternions, one can write [42] the generators of a two-dimensional irreducible representation (the defining representation, $\mathbf{2}_0$) as

$$e_1 = \frac{1}{2}(1 + i + j + k), \quad (76)$$

$$e_2 = \frac{1}{2}(1 + i + j - k). \quad (77)$$

The other $\mathbf{2}$ s are defined as $\mathbf{2}_a = \mathbf{2}_0 \otimes \mathbf{1}_a$. There is also a three-dimensional irreducible representation: $\mathbf{2}_0 \otimes \mathbf{2}_a = \mathbf{3} \oplus \mathbf{1}_a$. The 24 elements of $\widehat{\mathbb{E}}_6$ are

$$\left\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \right\}, \quad (78)$$

and the commutator subgroup $\widehat{\mathbb{E}}'_6 \simeq \widehat{\mathbb{D}}_2$ consists of the first eight elements in this list⁵.

The ordinary tetrahedral group \mathbb{E}_6 is a discrete subgroup of $SU(2)/\mathbb{Z}_2 \simeq SO(3)$. We now require that

$$e_1^3 = e_2^3 = (e_2 e_1)^2 = 1. \quad (79)$$

Only the one- and three-dimensional irreducible representations of $\widehat{\mathbb{E}}_6$ satisfy these relations. The two-dimensional irreducible representations correspond to including a non-trivial \mathbb{Z}_2 torsion in the previous relations. That is to say, they satisfy

$$e_1^3 = e_2^3 = (e_2 e_1)^2 = -1. \quad (80)$$

⁵This fact, coupled with the exact sequence Eq. (17), immediately gives the discrete torsion of $\widehat{\mathbb{E}}_6 \times \mathbb{Z}_n$ quoted in Ref. [20]. However, when n is even neither this group nor $\widehat{\mathbb{E}}_7 \times \mathbb{Z}_n$ is in $SU(3)$.

The representation theory of the binary octahedral group $\widehat{\mathbb{E}}_7$ proceeds along similar lines. Here, the presentation [21] is

$$e_1^4 = e_2^3 = (e_2 e_1)^2. \quad (81)$$

The one-dimensional irreducible representations are then

$$\mathbf{1}_0 : (e_1, e_2) = (+1, +1), \quad \mathbf{1}_1 : (e_1, e_2) = (-1, +1), \quad (82)$$

and the defining representation in terms of quaternions [42] is

$$e_1 = \frac{1}{\sqrt{2}}(1 + i), \quad (83)$$

$$e_2 = \frac{1}{2}(1 + i + j + k). \quad (84)$$

Tensor products yield two other **2**s, two **3**s, and a **4**. The 48 elements of $\widehat{\mathbb{E}}_7$ are

$$\left\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k), \frac{1}{\sqrt{2}}(\pm 1 \pm i), \frac{1}{\sqrt{2}}(\pm 1 \pm j), \right. \\ \left. \frac{1}{\sqrt{2}}(\pm 1 \pm k), \frac{1}{\sqrt{2}}(\pm i \pm j), \frac{1}{\sqrt{2}}(\pm i \pm k), \frac{1}{\sqrt{2}}(\pm j \pm k) \right\}, \quad (85)$$

and the commutator subgroup $\widehat{\mathbb{E}}'_7 \simeq \widehat{\mathbb{E}}_6$. The two **1**s, one of the **2**s (not the defining representation), and the two **3**s satisfy the relations of the ordinary octahedral group:

$$e_1^4 = e_2^3 = (e_2 e_1)^2 = 1. \quad (86)$$

The other irreducible representations have

$$e_1^4 = e_2^3 = (e_2 e_1)^2 = -1. \quad (87)$$

By the McKay Correspondence [43], the quivers of the binary polyhedral groups correspond to affine Dynkin diagrams of the exceptional Lie groups.

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